

# Zero-lag synchronization in coupled time-delayed piecewise linear electronic circuits

R. Suresh<sup>1</sup>, K. Srinivasan<sup>1</sup>, D. V. Senthilkumar<sup>2</sup>, I. Raja Mohamed<sup>3</sup>, K. Murali<sup>4</sup>, M. Lakshmanan<sup>1</sup>, and J. Kurths<sup>2,5,6</sup>

<sup>1</sup>*Centre for Nonlinear Dynamics, School of Physics,  
Bharathidasan University, Tiruchirapalli 620 024, India*

<sup>2</sup>*Potsdam Institute for Climate Impact Research, 14473 Potsdam, Germany*

<sup>3</sup>*Department of Physics, B. S. Abdur Rahman University, Chennai 600 048, India*

<sup>4</sup>*Department of Physics, Anna University, Chennai 600 025, India*

<sup>5</sup>*Institute of Physics, Humboldt University, 12489 Berlin, Germany*

<sup>6</sup>*Institute for Complex Systems and Mathematical Biology,  
University of Aberdeen, Aberdeen AB24 3UE, United Kingdom*

(Dated: February 1, 2013)

We investigate and report an experimental confirmation of zero-lag synchronization (ZLS) in a system of three coupled time-delayed piecewise linear electronic circuits via dynamical relaying with different coupling configurations, namely mutual and subsystem coupling configurations. We have observed that when there is a feedback between the central unit (relay unit) and at least one of the outer units, ZLS occurs in the two outer units whereas the central and outer units exhibit inverse phase synchronization (IPS). We find that in the case of mutual coupling configuration ZLS occurs both in periodic and hyperchaotic regimes, while in the subsystem coupling configuration it occurs only in the hyperchaotic regime. Snapshots of the time evolution of outer circuits as observed from the oscilloscope confirm the occurrence of ZLS experimentally. The quality of ZLS is numerically verified by correlation coefficient and similarity function measures. Further, the transition to ZLS is verified from the changes in the largest Lyapunov exponents and the correlation coefficient as a function of the coupling strength. IPS is experimentally confirmed using time series plots and also can be visualized using the concept of localized sets which are also corroborated by numerical simulations. In addition, we have calculated the correlation of probability of recurrence to quantify the phase coherence. We have also analytically derived a sufficient condition for the stability of ZLS using the Krasovskii-Lyapunov theory.

PACS numbers: 05.45.Xt, 05.45.Pq

Nonlinear time-delay systems constitute an important class of dynamical systems which are abundant in nature and in real life. Synchronizing such systems is having important implications in all areas of science. Different types of synchronization have been numerically identified in a variety of ensembles of time-delay systems. In contrast, only a very limited number of studies is available from experimental point of view. In this paper, we report a concrete experimental evidence of zero-lag synchronization (ZLS) in a system of three coupled time-delayed nonlinear electronic circuit with two different coupling configurations, namely mutual and subsystem coupling configurations. We have observed that when there is a feedback between the central and atleast one of the outer units, ZLS occurs in the two outer units whereas central and outer units exhibit inverse phase synchronization (IPS). The results are experimentally confirmed using snap shots of the time evolution, phase projection plots and the concept of localized sets along with corresponding numerical results. The transition to ZLS and IPS can be quantified from the changes in the largest Lyapunov exponents, correlation coefficient and correlation of probability of recurrence as a function of the coupling strength. We have also analytically deduced suf-

ficient stability conditions to confirm ZLS using the Krasovskii-Lyapunov theory.

## I. INTRODUCTION

In 2006, Fischer et al. reported zero-lag synchronization (ZLS) in two mutually delay coupled chaotic units via a relay unit (also known as isochronal synchronization) [1], in the absence of which, both systems exhibit lead or lag synchronization. Interestingly, this phenomenon could explain the occurrence of identical synchronization between widely separated cortical regions of the human brain despite of synaptic/dendritic delays [2–5]. ZLS has also been experimentally demonstrated in delay coupled semiconductor lasers [6], optoelectronic oscillators [7] and in low-dimensional delay coupled chaotic electronic circuits (without intrinsic time-delay) [8] via dynamical relaying. Recently Banerjee et al. reported that the coupling threshold for ZLS between the outermost identical oscillators decreases when an impurity (parameter mismatch) is introduced in the relay unit [9]. Further, synchronization condition as a function of Lyapunov exponents and parameters has been obtained [10]. ZLS has attracted a plethora of research activities, mainly because of its potential applications in secure communication over a public channel [11], and it was recently shown that it is possible to use ZLS phe-

nomenon in two mutually coupled chaotic systems for bidirectional secure communication (where both systems sending/receiving messages simultaneously at the same time) [12]. The above works clearly demonstrate the occurrence of ZLS in low-dimensional systems.

On the other hand, synchronization in coupled time-delay systems is an intriguing phenomenon because of the infinite-dimensional nature of the underlying systems which exhibit hyperchaotic attractors characterized by multiple positive Lyapunov Exponents (LEs) even for small values of time-delay. Synchronizing such time-delay systems is very challenging and has potential applications in diverse areas involving physical, chemical, biological, neurological and electrical systems [13–17]. Different types of synchronization have been recently observed numerically along with experimental evidence in coupled time-delay systems [18–22].

Motivated by the above works and ideas, in this paper, we report an experimental confirmation of ZLS via dynamical relaying in coupled nonlinear time-delayed electronic circuits in the hyperchaotic regime corroborated by numerical simulations. For this purpose, we have taken three identical time-delayed electronic circuits (exhibiting hyperchaotic attractors) and couple them with two different possible coupling configurations, namely mutual and subsystem coupling configurations. Similar coupling configurations have been previously employed in low-dimensional chaotic systems to achieve ZLS [23]. In these two coupling configurations, we find that when there is a feedback between the relay (central) system and at least one of the outer systems, ZLS takes place between the two outer systems, while the relay and outer systems exhibit inverse phase synchronization (IPS). Snapshots of the time evolution of the systems and phase projection plots as observed from the oscilloscope confirm the occurrence of ZLS in coupled time-delayed electronic circuits. Also the quality of synchronization can be numerically verified using the correlation coefficient and the similarity function. It is to be noted that in the case of mutual coupling configuration ZLS between the two outer systems occur both in the periodic and hyperchaotic regimes, whereas in the subsystem coupling configuration it occurs only in the hyperchaotic regime. These facts are characterized and confirmed from the changes in the largest LEs of the coupled systems and the correlation coefficient as a function of the coupling strength. We have also derived a sufficient stability condition for the occurrence of ZLS using the Krasovskii-Lyapunov functional theory. Further, IPS is experimentally confirmed using time series plots and phase coherence is characterized both qualitatively and quantitatively by the concept of localized sets and the Correlation of Probability of Recurrence (CPR), respectively.

The paper is organized as follows: In Sec. II, we describe the system employed to demonstrate the occurrence of ZLS and explain the circuit configuration. In Sec. III, we explain the existence of ZLS and characterize the results in mutual coupling configuration using

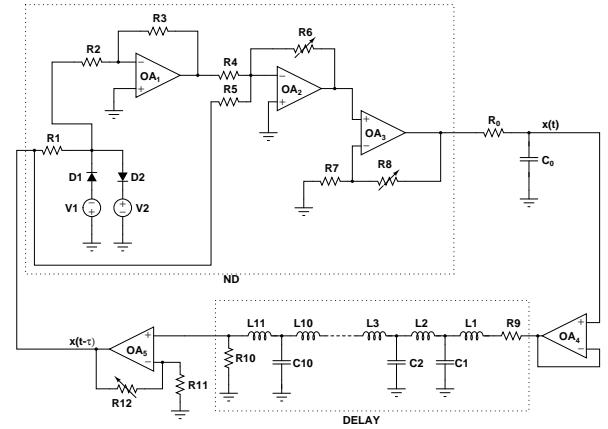


FIG. 1: Circuit diagram of a single time delayed feedback oscillator with a nonlinear device (ND) unit, a time-delay unit (DELAY) and a low pass first-order  $R_0C_0$  filter.

experimental and numerical evidences along with linear stability analysis. The occurrence of ZLS in subsystem coupling configuration is discussed in Sec. IV and finally Sec. V is dedicated to discussion and conclusion.

## II. SYSTEM DESCRIPTION

In this section, we briefly describe the system which we are using in this paper to demonstrate the synchronization phenomenon. We consider the following scalar delay differential equation

$$\dot{x} = -\alpha x(t) + \beta f(x(t - \tau)) \quad (1)$$

where  $\alpha, \beta$  are the system parameters, and  $\tau$  is the time-delay. The nonlinear function  $f(x)$  can be effectively implemented by a piecewise linear function defined as

$$f(x) = AF^* - Bx. \quad (2)$$

Here

$$F^* = \begin{cases} -x^*, & x < -x^* \\ x, & -x^* \leq x \leq x^* \\ x^*, & x > x^*. \end{cases} \quad (3)$$

The system parameters are chosen as  $\alpha = 1.0$ ,  $\beta = 1.2$ ,  $\tau = 6.0$ ,  $A = 5.2$ ,  $B = 3.5$  and  $x^*$  is the threshold value fixed at  $x^* = 0.7$ . These parameter values are used throughout the paper for numerical simulation. It may be noted that for the above values, the single system (1) exhibits a hyperchaotic attractor with multiple (four) positive LEs (see Fig. 6 in refs. [21, 24]).

### A. Circuit realization

The electronic circuit investigated here is given in Fig. 1 which describes the dynamics of Eq. (1) along

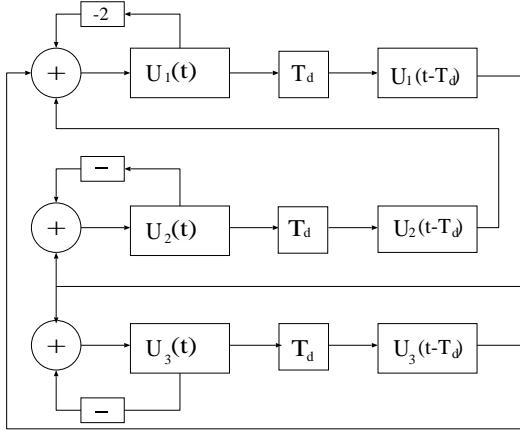


FIG. 2: Circuit block diagram of the three coupled time delayed feedback oscillator for mutual coupling configuration (4).

with the threshold nonlinear function  $f(x)$  given by Eq. (2). This circuit has a ring structure and consists of a diode based nonlinear device (ND) unit with two amplification stages (OA<sub>2</sub> and OA<sub>3</sub>), a time-delay unit (DELAY), and a low pass first order  $R_0C_0$  filter. In this circuit,  $\mu$ A741s are engaged as operational amplifiers. The constant voltage sources are  $V_1$  and  $V_2$ , and the voltage supply for all active devices is  $\pm 12V$ . One can adjust the threshold value of the three segment piecewise function (Eq. (3)) by altering the voltage values  $V_1$  and  $V_2$ . By applying Kirchhoff's laws to this circuit (Fig. 1), the state equation can be written as  $R_0C_0 \frac{dU(t)}{dt} = -U(t) + F[k_f(U(t-T_d))]$ , where  $U(t)$  is the voltage across the capacitor  $C_0$ ,  $U(t-T_d)$  is the voltage across the delay unit,  $T_d (= n\sqrt{LC})$  is the delay time,  $n$  is the number of LC filter units, and  $F[k_f(U(t-T_d))]$  is the static characteristic of the ND unit.

To analyze the above circuit equation, we transform it to the dimensionless oscillator (1) by defining the dimensionless variables and parameters as  $x(t) = \frac{U(t)}{U_s}$ ,  $t' = \frac{t}{R_0C_0}$ ,  $\tau = \frac{T_d}{R_0C_0}$ ,  $\alpha = 1.0$ ,  $k_f = \beta$ , and  $t' \rightarrow t$ . The circuit parameters are fixed as follows:  $R_1 = 1K\Omega$ ,  $R_2 = R_3 = 10K\Omega$ ,  $R_4 = 2K\Omega$ ,  $R_5 = 3K\Omega$ ,  $R_6 = 10.4K\Omega$  (trimmer-pot),  $R_7 = 1K\Omega$ ,  $R_8 = 5K\Omega$  (trimmer-pot),  $(R_9 = R_{10} = 1K\Omega$ ,  $R_{11} = 10K\Omega$ ,  $R_{12} = 20K\Omega$  (trimmer-pot),  $R_0 = 1.86K\Omega$ ,  $C_0 = 100nF$ ,  $L_i = 12mH$  ( $i = 1, 2, \dots, 11$ ),  $C_i = 470nF$  ( $i = 1, 2, \dots, 10$ ),  $n = 10$ .  $T_d = 0.751ms$ ,  $R_0C_0 = 0.268ms$ , and so the time-delay  $\tau \approx 2.8$  for the chosen values of the circuit parameters.

The bifurcation scenario and the dynamics of this time-delayed chaotic oscillator has been investigated in some detail in Ref. [24]. In the following sections, we will demonstrate the existence of zero lag synchronization in coupled time-delayed piecewise linear electronic circuits of the above form for two different coupling configurations.

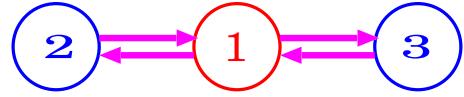


FIG. 3: (Color online) Schematic diagram for mutual coupling configuration.

### III. MUTUAL COUPLING CONFIGURATION

In this section, we demonstrate the occurrence of ZLS in coupled time-delay systems with mutual coupling configuration. For this purpose, we have considered three identical time-delayed electronic circuits each of the form of Fig. 1 and coupled them as shown in the block diagram of Fig. 2. The state equations for the coupled electronic circuit (Fig. 2) can be written as

$$R_0C_0 \frac{dU_1(t)}{dt} = -U_1(t) + f[k_f(U_1(t-T_d))] + \varepsilon'[U_2(t-T_d) - 2U_1(t) + U_3(t-T_d)], \quad (4a)$$

$$R_0C_0 \frac{dU_2(t)}{dt} = -U_2(t) + f[k_f(U_2(t-T_d))] + \varepsilon'[U_1(t-T_d) - U_2(t)], \quad (4b)$$

$$R_0C_0 \frac{dU_3(t)}{dt} = -U_3(t) + f[k_f(U_3(t-T_d))] + \varepsilon'[U_1(t-T_d) - U_3(t)], \quad (4c)$$

where the variables  $U_1(t)$ ,  $U_2(t)$  and  $U_3(t)$  correspond to the output variables of each circuit. By defining the normalized variables and parameters as given in Sec. II A and  $\varepsilon' = \varepsilon$ , one obtains the equivalent dimensionless equation as follows:

$$\dot{x}_1(t) = -\alpha x_1(t) + \beta f(x_1(t-\tau)) + \varepsilon[x_2(t-\tau) - 2x_1(t) + x_3(t-\tau)], \quad (5a)$$

$$\dot{x}_2(t) = -\alpha x_2(t) + \beta f(x_2(t-\tau)) + \varepsilon[x_1(t-\tau) - x_2(t)], \quad (5b)$$

$$\dot{x}_3(t) = -\alpha x_3(t) + \beta f(x_3(t-\tau)) + \varepsilon[x_1(t-\tau) - x_3(t)]. \quad (5c)$$

Here,  $x_1(t)$  is the central relay system,  $x_2(t)$  and  $x_3(t)$  are the two outer systems and  $\varepsilon$  is the coupling strength between the systems.

In this coupling configuration, the relay unit (system 1) sends its delayed signal to the outer units (systems 2 and 3) and both the outer units also send their delayed feedback to the relay unit (Fig. 2). For simple illustration, the schematic diagram for this coupling configuration is shown in Fig. 3.

#### A. Experimental and numerical results

In the absence and for lower values of coupling strength, the three systems evolve freely according to

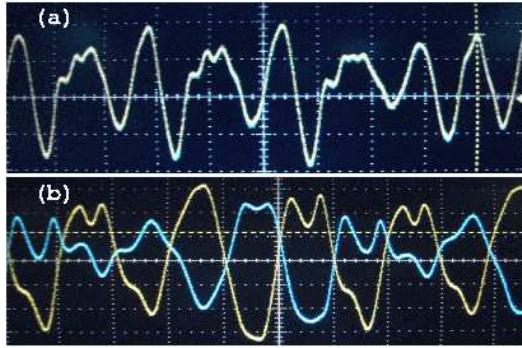


FIG. 4: (Color online) Experimental verification of ZLS and IPS in mutual coupling configuration. (a) Time evolution of both the outer circuits (2 and 3) displays ZLS ( $U_2(t)$  - yellow curve and  $U_3(t)$  - blue curve); vertical scale 1 unit = 2V, horizontal scale 1 unit = 1ms and (b) time evolution of the relay and one of the outer circuits shows IPS ( $U_1(t)$  - yellow curve and  $U_2(t)$  - blue curve); vertical scale 1 unit = 2V, horizontal scale 1 unit = 2ms.

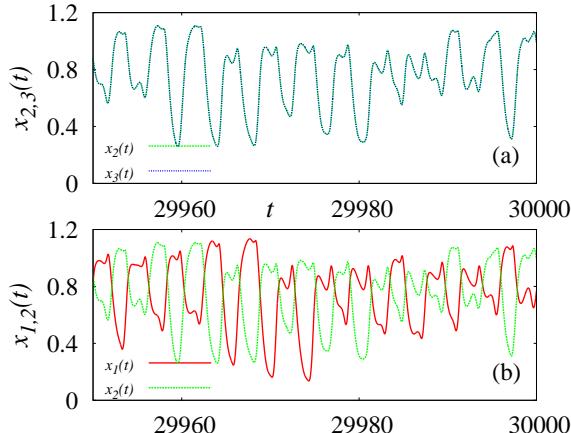


FIG. 5: (Color online) Numerically obtained time series of the systems in mutual coupling configuration (Eq. (5)) for  $\varepsilon = 1.55$ . (a) The two outer systems  $x_2(t)$  and  $x_3(t)$  display ZLS, (b) the outer system  $x_2(t)$  shows IPS with the relay system  $x_1(t)$ .

their own dynamics. For sufficiently large values of coupling strength, if the two outer systems are directly connected without the relay unit (system 1), they get synchronized with a lead or lag time equal to the coupling delay [8, 25, 26]. On the other hand, if we couple them through the relay system (as depicted in Fig. 3), then remarkably both the outer systems are synchronized without a time lag (zero-lag). In addition, the outer systems (2 and 3) exhibit IPS with the relay system.

Snapshots of wave forms of the circuits as seen in the oscilloscope are shown in Fig. 4. ZLS between the two outer circuits is presented in Fig. 4(a) and the realization of IPS between one of the outer and the central circuits is given in Fig. 4(b). Figure 5 shows the numerical sim-

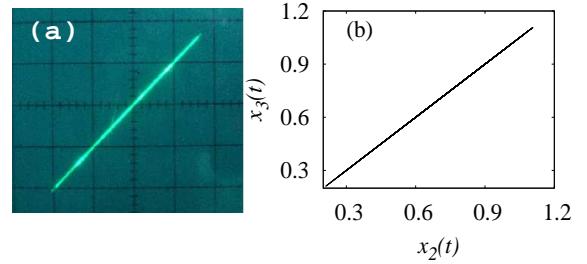


FIG. 6: (Color online) (a) Experimental (x-axis: voltage  $U_2(t)$  (1 unit = 2.0 V); y-axis:  $U_3(t)$  (1 unit = 2.0 V)) and (b) numerical realization of the phase portraits of the systems  $x_2(t)$  and  $x_3(t)$  showing ZLS.

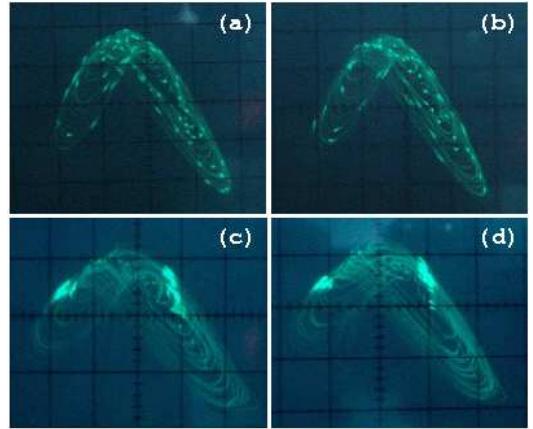


FIG. 7: (Color online) Experimental realization of the framework of localized sets for mutual coupling configuration. (a), (b) The sets are spread over the attractors indicating the absence of phase coherence for lower values of coupling strength. (c), (d) For a sufficiently large value of coupling strength, the sets are localized on the attractors which indicates phase coherence. In (a), (c) x-axis: voltage  $U_1(t)$  (1 unit = 0.5 V), y-axis: voltage  $U_1(t - T_d)$  (1 unit = 2.0 V) and in (b), (d) x-axis: voltage  $U_2(t)$  (1 unit = 0.5 V), y-axis: voltage  $U_2(t - T_d)$  (1 unit = 2.0 V).

ulation of Eq. (5) for the coupling strength  $\varepsilon = 1.55$ . Systems 2 and 3 maintain a perfect ZLS as depicted in Fig. 5(a). ZLS is further confirmed from the phase portraits of the corresponding systems. Figures 6(a) and 6(b) show experimental and numerical realizations of the phase portraits of the two outer systems. The diagonal line represents the existence of complete (zero-lag) synchronization between the two outer systems 2 and 3. Further, the outer system 2 is in IPS with the relay system 1, where essentially the extrema of the time series of the two systems occur opposite to each other as clearly displayed in Figs. 4(b) and 5(b).

A qualitative measurement of phase coherence can be visualized both experimentally and numerically by using the framework of localized sets [27]. The basic idea of this characterization is that the set of points obtained by

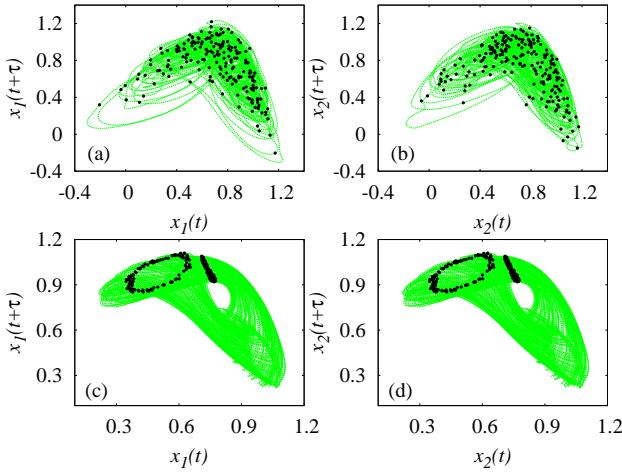


FIG. 8: (Color online) Numerically obtained equivalent figures for Fig. 7. (a), (b) correspond to the absence of phase coherence for  $\varepsilon = 0.5$ . In (c), (d) the sets are localized on the attractors indicating phase locking for  $\varepsilon = 1.55$ .

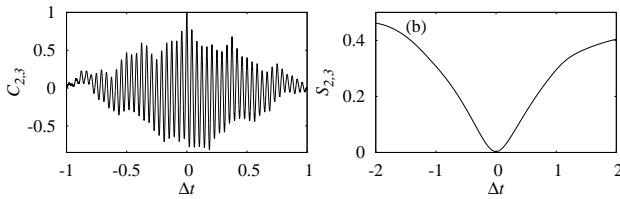


FIG. 9: ZLS of the two outer systems 2 and 3 are confirmed by (a) the correlation coefficient (Eq. 6) and (b) the similarity function (Eq. 7) for  $\varepsilon = 1.55$ .

sampling the time series of the system 2 whenever a maximum occurs in the relay system is plotted along with the attractor of the system 2 and vice-versa. Depending upon the property of the set, one can find whether phase coherence exists or not. The coupled systems are said to be phase synchronized upon localization of the observed sets on the attractor. On the other hand, the sets that spread over the entire attractor confirm asynchronization. This approach provides a general way to identify phase synchronization even in non-phase-coherent attractors.

For the experimental implementation of the framework of localized sets, maxima of the outer system and minima of the relay units are taken as reference. Using the circuit given in Fig. 7.12 in pp. 147 of Ref. [28], we generate the impulse whenever the input signal of the outer (relay) system attains maximum (minima). While the attractor of the relay (outer) system is in the X-Y channel of the oscilloscope, we feed the impulse signal to Z- input. Whenever the impulse hits the attractor of the relay (outer) unit one can see the bright spot on the attractor of the relay (outer) system. As pointed out above, if the both systems are in CPS the spots (sets) are localized on the attractors otherwise the sets are spread over the

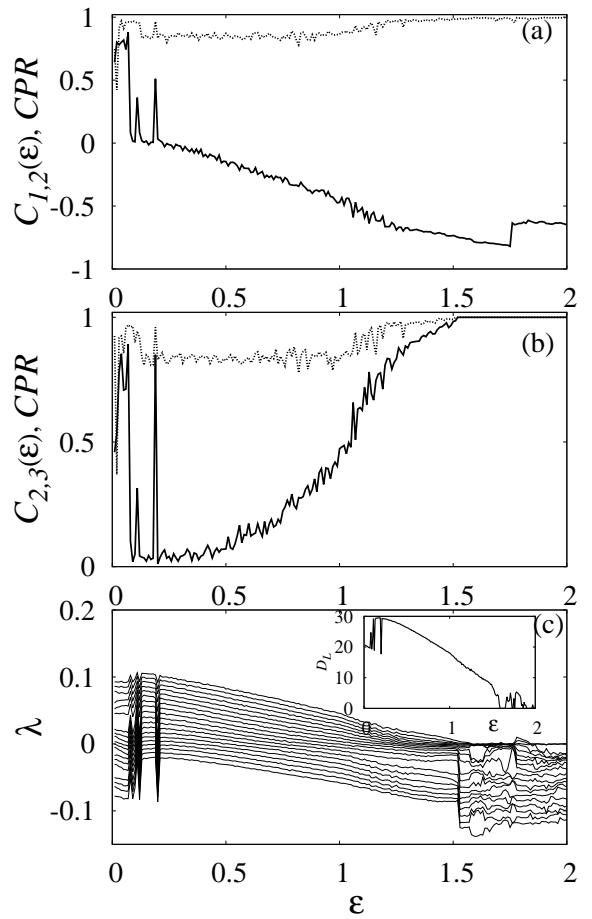


FIG. 10: (a) Correlation coefficient  $C_{1,2}(\varepsilon)$  (continuous curve) and the index CPR (dotted curve) of the systems 1, 2, (b) correlation coefficient  $C_{2,3}(\varepsilon)$  and the CPR of the two outer systems 2, 3 and (c) display the transition in the largest LEs of the coupled systems as a function of the coupling strength in mutual coupling configuration. The inset figure in (c) shows the Kaplan-Yorke dimension ( $D_L$ ) as a function of the coupling strength.

entire attractor.

Figures 7 (a) and 7 (b) show the experimentally obtained attractors along with the sets of the relay system and the system 2, respectively. Here the sets are distributed over the entire attractor for a low value of coupling strength related to the absence of phase synchronization. The corresponding numerically obtained figures are plotted in Figs. 8 (a) and 8 (b) for the value of coupling strength  $\varepsilon = 0.5$ . If we increase the coupling strength to sufficiently large value, the sets are localized on the attractor as depicted in Figs. 7 (c) and 7 (d) confirming the phase locking of both systems. The corresponding numerical figures are plotted in Figs. 8 (c) and 8 (d) for the value of  $\varepsilon = 1.55$ .

In order to characterize the quality of synchronization and to find out the lag between the time series of the two outer systems, we study the correlation coefficient

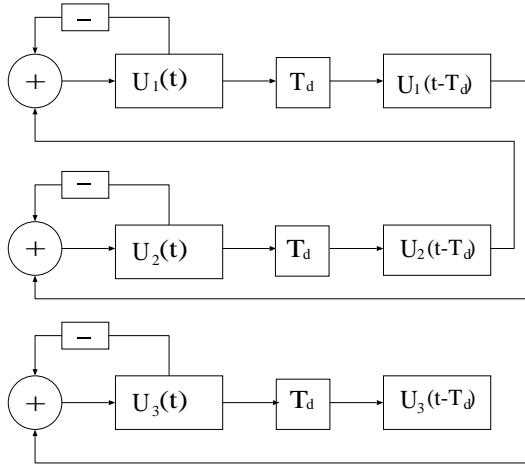


FIG. 11: Circuit block diagram of the three coupled time delayed feedback oscillator for the subsystem coupling configuration (14).

and the similarity function between the outer systems. The correlation coefficient and the similarity function, respectively, are given by the expressions

$$C_{2,3} = \frac{\langle (x_2(t) - \langle x_2(t) \rangle)(x_3(t + \Delta t) - \langle x_3(t) \rangle) \rangle}{\sqrt{\langle (x_2(t) - \langle x_2(t) \rangle)^2 \rangle \langle (x_3(t) - \langle x_3(t) \rangle)^2 \rangle}}, \quad (6)$$

$$S_{2,3} = \frac{\langle (x_2(t + \tau) - x_2(t))^2 \rangle}{\sqrt{\langle x_2^2(t) \rangle \langle x_3^2(t) \rangle}}, \quad (7)$$

where the  $\langle \rangle$  brackets indicate time averaging. Using Eq. (6), we compute the correlation between the outer systems as a function of time shifts ( $\Delta t$ ), characterizing the quality of the synchronization. The time lag between the two outer systems can be determined by the position of the global maximum of the correlation coefficient for ZLS. The global maximum of the correlation coefficient has a value close to unity, at  $\Delta t = 0$ , which reflects that there is no delay between both the outer systems. In the case of the similarity function (Eq. (7)), if  $x_2(t) = x_3(t)$  (for ZLS), then the similarity function reaches the minimum  $S \approx 0$  for time shift  $\tau = 0$ . For a nonzero value of the time shift  $\tau$ ,  $S \approx 0$  corresponds to a lag between the two signals  $x_2(t)$  and  $x_3(t)$ . Figure 9 (a) shows the correlation coefficient of the systems 2 and 3 as a function of  $\Delta t$  with its global maximum ( $C_{2,3} \approx 1.0$ ) at  $\Delta t = 0$  confirming the ZLS between them for the coupling strength  $\varepsilon = 1.55$ . Similarly, the similarity function of the two outer systems 2 and 3 is depicted in Fig. 9 (b) which clearly indicates that the minimum  $S_{2,3} \approx 0$  occurs at  $\tau = 0$  again confirming the existence of ZLS between the systems  $x_2(t)$  and  $x_3(t)$ .

The transition from asynchronization to ZLS between the outer systems is further characterized by calculating the correlation coefficient (with  $\Delta t = 0$  in Eq. (6)) and the transition in the largest LEs of the coupled time-delay systems as a function of the coupling strength in the range  $\varepsilon \in (0, 2)$ . Further, phase coherence is

characterized by using one of the recurrence quantifications, namely the Correlation of Probability of Recurrence (CPR) [29, 30]. If the phases of the coupled systems are entrained (mutually locked) completely, then the probability of the recurrence is maximal at a time and  $\text{CPR} \approx 1$ . In contrast, one can expect a drift in the probability of recurrences resulting in low values of CPR characterizing the degree of locking between the coupled systems [29, 30]. Figures 10(a) and 10(b) display the correlation coefficient (continuous curve) and the index CPR (dotted curve) between the systems 1, 2 and systems 2, 3, respectively. This way, we can distinguish the following regimes:

(i) For certain low values of  $\varepsilon$ , all three systems exhibit in-phase synchronization where  $\text{CPR} \approx 1$ . Further, the high degree of the correlation coefficients  $C_{1,2}(\varepsilon)$  and  $C_{2,3}(\varepsilon)$  but with values less than unity quantifies the degree of correlation in their amplitudes, corroborating the existence of approximate complete synchronization between all the three systems. This synchronization transition is also confirmed by the changes in the largest LEs of the coupled systems (5). In Fig. 10(c), we have plotted the 19 largest LEs as a function of the coupling strength  $\varepsilon$ . For  $\varepsilon = 0$ , there are 12 positive LEs (each system has four positive LEs). We note that for nonzero low values of  $\varepsilon$  there exists only 9 positive LEs confirming the transition to the approximate synchronization with three of the positive LEs becoming negative in the above range of  $\varepsilon$ .

(ii) In the intermediate range of  $\varepsilon$ ,  $C_{1,2}$  decreases and  $C_{2,3}$  increases upon increasing  $\varepsilon$  indicating the loss of correlation between the relay and outer units, while the index CPR oscillates around the value 0.8 among all the three systems quantifying the degree of drift in their phases. On the other hand, the degree of correlation between the outer units increases at the same time leading to the transition to ZLS from approximate complete synchronization. Further, the largest Lyapunov exponents gradually acquire values less than zero, which also confirms the above synchronization transition.

(iii) If we increase the coupling strength further, the index CPR of the systems 1, 2 increases and reaches almost unity for  $\varepsilon > 1.4$ , whereas their correlation coefficient  $C_{1,2}(\varepsilon)$  becomes negative, which indeed confirms the inverse phase synchronization between units 1 and 2 [Fig. 10(a)]. On the other hand, the correlation coefficient  $C_{2,3}(\varepsilon)$  and the index CPR of the two outer systems increase as a function of  $\varepsilon$  and for  $\varepsilon = 1.51$  both reach the unit value corroborating the occurrence of ZLS between them [Fig. 10(b)].

(iv) Further, except for the largest three positive LEs, all the other LEs of the coupled system become negative for  $\varepsilon > 1.51$  confirming the existence of ZLS between the outer units in the chaotic regime, which is the common synchronization manifold arising due to mutual interaction among all the systems. We also note that there exist windows of ZLS in periodic regimes, where all the LEs are less than zero. It is worth to emphasize at this point

that we have observed ZLS where the synchronization manifold is both periodic and chaotic depending on the value of the coupling strength  $\varepsilon$  as unveiled by the Lyapunov exponents. Further, the Kaplan-Yorke dimension ( $D_L$ ) [31, 32] is depicted in the inset in Fig. 10(c) (defined as the sum of all the positive LEs) as a function of the coupling strength, which clearly indicates the periodic ( $D_L = 0$ ) and chaotic regimes ( $D_L > 0$ ).

### B. Linear stability analysis for ZLS

In this subsection, we analytically investigate the existence of ZLS between the two outer oscillators in the mutual coupling configuration [Eq. (5)(b, c)]. For this, we consider the time evolution of the difference system with the state variable  $\Delta = x_2(t) - x_3(t)$  (which corresponds to the ZLS manifold of the two outer systems) for small values of  $\Delta$ . The evolution equation for the synchronization manifold can be written as

$$\dot{\Delta} = -(\alpha + \varepsilon)\Delta + \beta[AF^{*'}(x_2(t - \tau)) - B]\Delta_\tau, \quad \Delta_\tau = \Delta(t - \tau) \quad (8)$$

so that the stability condition can be deduced analytically. The synchronization manifold is locally attracting if the origin of (8) is stable. From the Krasovskii-Lyapunov theory [33], one can define a continuous positive definite Lyapunov functional of the form

$$V(t) = \frac{1}{2}\Delta^2 + \mu \int_{-\tau}^0 \Delta^2(t + \theta) d\theta, \quad V(0) = 0, \quad (9)$$

where  $\mu$  is an arbitrary positive parameter  $\mu > 0$ . The above Lyapunov function  $V(t)$  approaches zero as  $\Delta \rightarrow 0$ . The derivative of the above equation (9) along the trajectory of the synchronization manifold (8),

$$\frac{dV}{dt} = -[\alpha + \varepsilon]\Delta^2 + \beta[AF^{*'}(x_2(t - \tau)) - B]\Delta\Delta_\tau + \mu\Delta^2 - \mu\Delta_\tau^2, \quad (10)$$

should be negative for the stability of the synchronization manifold  $\Delta = 0$ . This requirement results in the condition for stability as

$$(\alpha + \varepsilon) > \frac{\beta^2}{4\mu}[AF^{*'}(x_2(t - \tau)) - B]^2 + \mu = \Phi(\mu). \quad (11)$$

One may note that  $\Phi(\mu)$  as a function of  $\mu$  for a given  $F^{*'}(x)$  has an absolute minimum at  $\mu = |\beta[AF^{*'}(x_2(t - \tau)) - B]|/2$  with  $\Phi_{min} = |\beta[AF^{*'}(x_2(t - \tau)) - B]|$ . Since  $\Phi \geq \Phi_{min} = |\beta[AF^{*'}(x_2(t - \tau)) - B]|$ , from the inequality (11), a sufficient condition for the asymptotic stability is

$$(\alpha + \varepsilon) > |\beta(AF^{*'}(x_2(t - \tau)) - B)|. \quad (12)$$

Now, from the form of the piecewise linear function  $f(x)$  in Eq. (3), we have,

$$|F^{*'}(x_2(t - \tau))| = \begin{cases} 0, & |x| > x^* \\ 1, & |x| \leq x^* \end{cases} \quad (13)$$

Consequently, the stability condition (12) becomes  $(\alpha + \varepsilon) > |\beta(A - B)|$ , corresponding to the inner regime  $|x| \leq x^*$  where most of the dynamics is confined, for the asymptotic synchronized state  $\Delta = 0$ . From our extensive numerical analysis as noted above, we find that ZLS between the two outer oscillators occur for the coupling strength  $\varepsilon = 1.51$  satisfying the stability condition  $\varepsilon > |\beta(A - B)| - \alpha = 1.04$ . However, we also note that the stability condition corresponding to the outer regime  $|x| > x^*$ , that is,  $(\alpha + \varepsilon) > |\beta B|$  is not validated by our numerical results in achieving ZLS, essentially because the stability condition obtained from the Lyapunov functional theory is only a sufficiency condition. Moreover, the role of the relay unit is not captured by the evolution equation (8) for the ZLS manifold in the above stability analysis.

### IV. SUBSYSTEM COUPLING CONFIGURATION

Next we consider the coupling configuration, where the relay system sends its delayed signal to the two outer systems and only one system (here system 2) sends its delayed feedback to the relay system. This configuration is called a subsystem coupling. The circuit for the subsystem coupling configuration is shown in Fig. 11 as a block diagram. The state equations for the coupled electronic circuit (Fig. 11) can be written as follows:

$$R_0 C_0 \frac{dU_1(t)}{dt} = -U_1(t) + f[k_f U_1(t - T_d)] + \varepsilon'[U_2(t - T_d) - U_1(t)], \quad (14a)$$

$$R_0 C_0 \frac{dU_2(t)}{dt} = -U_2(t) + f[k_f U_2(t - T_d)] + \varepsilon'[U_1(t - T_d) - U_2(t)], \quad (14b)$$

$$R_0 C_0 \frac{dU_3(t)}{dt} = -U_3(t) + f[k_f U_3(t - T_d)] + \varepsilon'[U_1(t - T_d) - U_3(t)]. \quad (14c)$$

The equivalent dimensionless equations of motion (see Sec. II) for the above configuration can be given as

$$\dot{x}_1(t) = -\alpha x_1(t) + \beta f(x_1(t - \tau)) + \varepsilon[x_2(t - \tau) - x_1(t)], \quad (15a)$$

$$\dot{x}_2(t) = -\alpha x_2(t) + \beta f(x_2(t - \tau)) + \varepsilon[x_1(t - \tau) - x_2(t)], \quad (15b)$$

$$\dot{x}_3(t) = -\alpha x_3(t) + \beta f(x_3(t - \tau)) + \varepsilon[x_1(t - \tau) - x_3(t)]. \quad (15c)$$

The schematic diagram for this configuration is sketched in Fig. 12. In the absence of the coupling ( $\varepsilon = 0$ ), all the three systems evolve independently and therefore there is no synchronization between them. As the coupling strength increases ( $\varepsilon > 0$ ) the two outer systems achieve ZLS, while exhibiting IPS with the relay system.

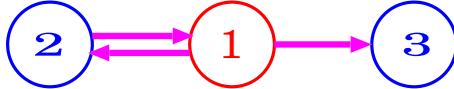


FIG. 12: (Color online) Schematic diagram for the subsystem coupling configuration.

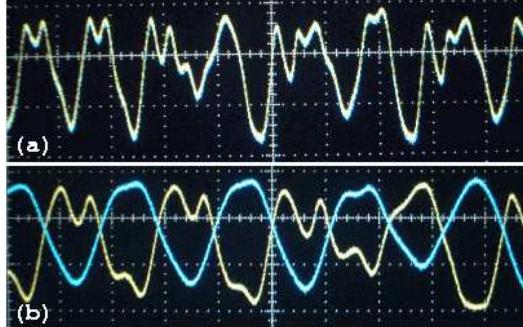


FIG. 13: (Color online) Experimental verification of ZLS and IPS in subsystem coupling configuration. (a) Time evolution of both the outer circuits ( $U_2(t)$  - yellow curve and  $U_3(t)$  - blue curve) displaying ZLS; x-axis: 1 unit -  $2ms$ , y-axis: 1 unit -  $2V$  and (b) time series of the relay and one of the outer circuits showing IPS (yellow curve  $U_1(t)$  and blue curve  $U_2(t)$ ). x-axis: 1 unit -  $2ms$ , y-axis: 1 unit -  $2V$ .

The experimental wave forms for suitable  $\varepsilon$  is shown in Fig. 13, where the time evolution of the two outer circuits is displayed in Fig. 13(a) exhibiting ZLS. The time evolution of the relay and one of the outer circuits (1 and 2) is displayed in Fig. 13(b) confirming IPS. The time evolution of all the three systems is also obtained using numerical simulations and depicted in Fig. 14 for the value of coupling strength  $\varepsilon = 1.6$ . Figure 14(a) shows the time traces of the two outer systems, displaying ZLS. The occurrence of IPS between the relay and outer unit (systems 1 and 2) can also be clearly seen in Fig. 14(b) where the maxima of the time series of the two systems are exactly opposite to each other. Figures 15 (a) and 15 (b) show the phase portraits (obtained by experimental and numerical simulations, respectively) of the outer systems (2 and 3) which display ZLS.

The phase coherence between the outer and the relay systems can again be visualized by plotting the localized sets. Figure 16 shows the experimentally obtained attractors along with the sets of the relay system and the system 2. In Figs. 16 (a) and 16 (b) the sets are distributed over the entire attractor for lower values of coupling strength  $\varepsilon$  due to the absence of phase synchronization. The corresponding numerically obtained figures are plotted in Figs. 17 (a) and 17 (b) for  $\varepsilon = 0.5$ . If we increase the coupling strength to a sufficiently large value, the sets are localized on the attractor as depicted in Figs. 16 (c) and 16 (d) confirming the phase locking of both the systems. The corresponding numerically obtained figures are plotted in Figs. 17 (c) and 17 (d) for the value of  $\varepsilon = 1.6$ .

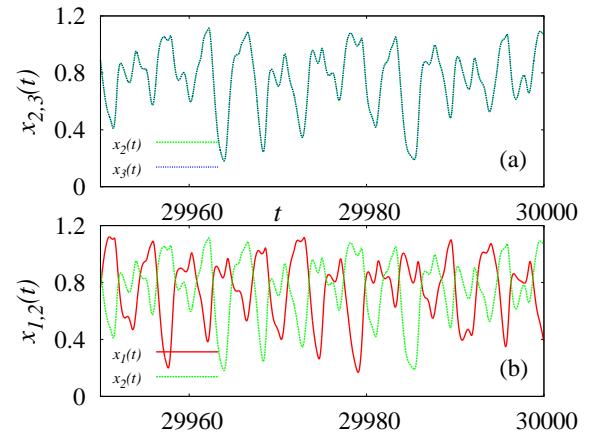


FIG. 14: (Color online) Numerically obtained time series of the systems in subsystem coupling configuration (Eq. (15)) for  $\varepsilon = 1.6$ . (a) The two outer systems  $x_2(t)$  and  $x_3(t)$  display ZLS, while (b) the outer system  $x_2(t)$  shows IPS with the relay system  $x_1(t)$ .

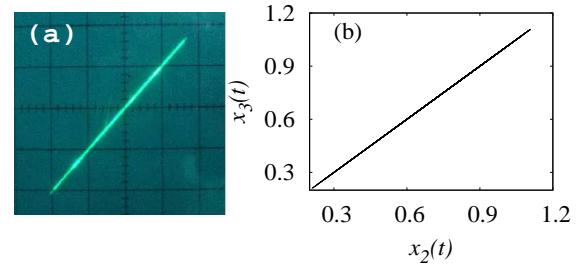


FIG. 15: (Color online) (a) Experimental (x-axis: voltage  $U_2(t)$  (1 unit =  $2.0$  V), y-axis: voltage  $U_3(t)$  (1 unit =  $2.0$  V)) and (b) numerical realization of the phase portraits of the systems  $x_2(t)$  and  $x_3(t)$  in subsystem coupling configuration showing ZLS.

As before, the existence of ZLS is further characterized and confirmed by calculating the correlation coefficient (6) and the similarity function (7) of the systems 2 and 3. The global maximum of the correlation of the systems 2 and 3 has a value close to unity ( $C_{2,3} \approx 0.99$ ) at  $\Delta t = 0$  indicating a complete synchronized behavior and that there is no lag between the two systems (Fig. 18(a)) for  $\varepsilon = 1.6$ . Figure 18(b) shows the minimum of the similarity function  $S_{2,3} \approx 0$  at  $\tau = 0$ , which also indicates that both the systems exhibit ZLS.

The transition from asynchronization to ZLS of the two outer systems are again characterized by calculating the correlation coefficient (with  $\Delta t = 0$  in Eq. (6)), the transition in LEs, while the phase coherence can be quantified by using the index CPR as a function of the coupling strength  $\varepsilon \in (0, 2)$ . Figures 19(a) and 19(b) depict the correlation coefficient (continuous curve) and the index CPR (dotted curve) between the systems 1, 2

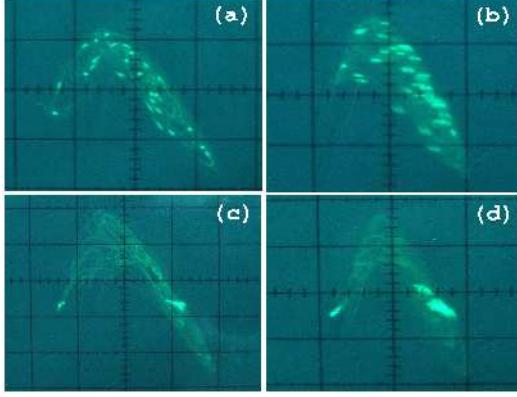


FIG. 16: (Color online) Experimental realization of the framework of localized sets for subsystem coupling configuration. (a), (b) The sets are spread over the entire attractors indicating the absence of phase coherence for lower values of coupling strength. (c), (d) For sufficiently large coupling value the sets are localized on the attractors which indicates the phase locking. In (a), (c) x-axis: voltage  $U_1(t)$  (1 unit = 0.5 V), y-axis: voltage  $U_1(t - T_d)$  (1 unit = 2.0 V) and in (b), (d) x-axis: voltage  $U_2(t)$  (1 unit = 0.5 V), y-axis: voltage  $U_2(t - T_d)$  (1 unit = 2.0 V).

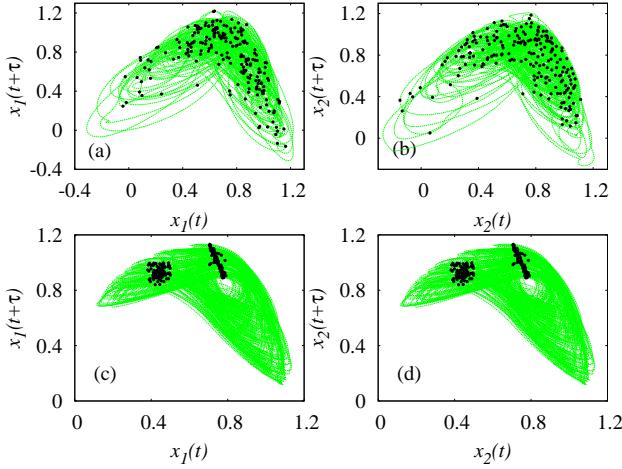


FIG. 17: (Color online) Numerically obtained equivalent figures for Fig. 16. (a), (b) correspond to the absence of phase coherence for  $\varepsilon = 0.5$ , (c), (d) the sets are localized on the attractors related to the occurrence of IPS for  $\varepsilon = 1.6$ .

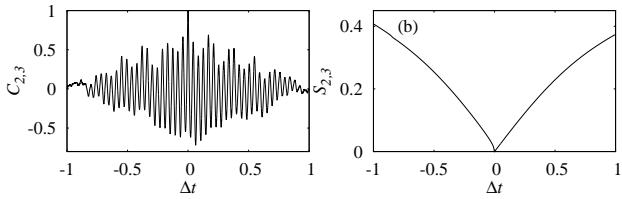


FIG. 18: Confirmation of ZLS of the two outer systems by (a) correlation coefficient and (b) the similarity function for  $\varepsilon = 1.6$  in subsystem coupling configuration.

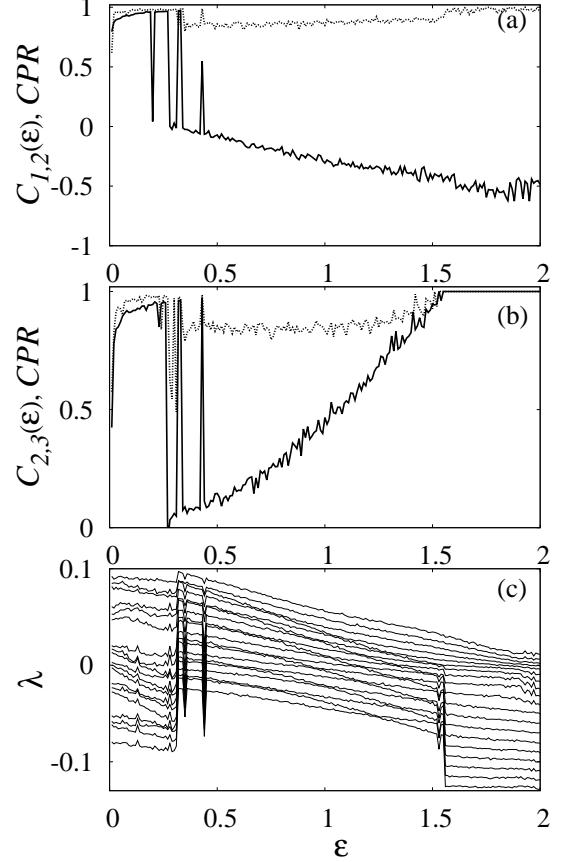


FIG. 19: (a) Correlation coefficient  $C_{1,2}(\varepsilon)$  and the index CPR of the systems 1, 2, (b) correlation coefficient  $C_{2,3}(\varepsilon)$  and the CPR of the two outer systems 2, 3 and (c) display of the transition of the largest LEs of the coupled systems as a function of the coupling strength in subsystem coupling configuration.

and 2, 3, respectively, as a function of  $\varepsilon$  exhibiting the following regimes:

(i) For certain lower values of coupling strength ( $\varepsilon < 0.25$ ), CPR reaches the unit value while the correlation coefficients  $C_{1,2}(\varepsilon)$  and  $C_{2,3}(\varepsilon)$  reach values near to unity (but not exactly 1), which indicate that all the three systems exhibit in-phase synchronization (approximate complete synchronization) [Figs. 19(a) and 19(b)]. This transition is also confirmed from the changes in the LEs of the coupled systems where they become negative for the corresponding values of  $\varepsilon$  [Fig. 19(c)].

(ii) Beyond  $\varepsilon > 0.25$  there is a desynchronization transition, where the CPR and correlation coefficient drop down well below unity and some of the positive LEs gradually acquire negative values.

(iii) If we continue to increase the coupling further, the CPR of the systems 1, 2 increases and reaches the unit value at  $\varepsilon \approx 1.6$ , whereas the correlation coefficient  $C_{1,2}(\varepsilon)$  becomes negative as a function of  $\varepsilon$ , which also confirms the onset of inverse phase synchronization

between systems 1 and 2 [Fig. 19(a)]. But the correlation coefficient ( $C_{2,3}(\varepsilon)$ ) and the index CPR of the two outer systems 2, 3 increase as a function of the coupling strength and for  $\varepsilon = 1.55$ , both the measures reach the unit value indicating the existence of persistent ZLS between the systems 2 and 3 [Fig. 19(b)].

(iv) The above synchronization transition is further confirmed by the transition in the largest LEs of the coupled systems (15). In Fig. 19(c) we plot nineteen largest LEs (with nine positive LEs). The first three LEs correspond to the largest LEs of the three independent systems. If we increase  $\varepsilon$ , the largest LEs of the outer systems (second and third largest LEs) become negative at  $\varepsilon \approx 1.55$  indicating the occurrence of ZLS in the subsystem coupling configuration.

(v) It is also to be noted that in the subsystem coupling configuration there always exist multiple positive LEs confirming the existence of ZLS in the hyperchaotic regime [Fig. 19(c)]. In contrast to the case of mutual coupling, there exists no periodic regime because one of the subsystems remains unaffected and ZLS occurs only in hyperchaotic regime. In addition, all the systems in the subsystem coupling configuration exhibit in-phase and approximate CS over a large range of  $\varepsilon$  for lower coupling strengths compared to the occurrence of a very few spikes of in-phase and approximate CS in the mutual coupling configuration.

Finally, to carry out a linear stability analysis for the subsystem coupling configuration (15), again we consider the difference between the two outer systems  $\Delta = x_2(t) - x_3(t)$ . The error equation corresponding to the synchronization manifold,  $\dot{\Delta} = -(\alpha + \varepsilon)\Delta + \beta[Af'(x_2(t - \tau)) - B]\Delta_\tau$ , which is exactly same as Eq. (8) in Sec. III B. So the sufficient condition for the stability of ZLS is also the same as in Eq. (12). Also from the numerical results we find that ZLS between the two outer systems occur for  $\varepsilon = 1.55$ , which satisfies the condition  $(\alpha + \varepsilon) > |\beta(A - B)| \approx (1.55 > 1.04)$ .

## V. CONCLUSION

We have reported here the first experimental confirmation of ZLS in a system of three coupled identical piecewise linear time-delayed electronic circuits via dynamical relaying with different coupling configurations, namely mutual and subsystem coupling configurations. From the obtained results, we have identified that when there is a feedback between the central and at least one

of the outer systems, occurrence of ZLS in the two outer systems takes place, while the central and outer systems exhibit IPS. In the above two cases, the central unit plays a key role to obtain ZLS. In the case of mutual coupling configuration ZLS occurs in both periodic and hyperchaotic regimes, while in the subsystem coupling configuration it occurs only in the hyperchaotic regime. We also find that for certain lower values of the coupling strength all the three systems exhibit in-phase synchronization and for large coupling (here  $\varepsilon > 1.5$ ) both outer systems exhibit ZLS, while the central and outer systems are in IPS. The results are experimentally confirmed from snapshots of the time evolution and phase projections of the outer systems. The quality of synchronization is numerically verified using the correlation coefficient and the similarity function. The transition to ZLS is characterized and confirmed from the changes in the largest LEs and the correlation coefficient of the coupled systems as a function of the coupling strength. Further, IPS is experimentally confirmed using snapshots of time series plots and the phase coherence is characterized both qualitatively and quantitatively by using the concept of localized sets and the index CPR, respectively. We have also analytically derived the stability condition to confirm the occurrence of ZLS using the Krasovskii-Lyapunov theory. One may also add that even though we have considered only a system of three coupled oscillators, one can treat each of the oscillators as corresponding to one group of synchronized oscillators. In fact, one can consider such groups of oscillators with different topologies to explain neuronal information processing among different parts of the brain, synchronization among groups of birds or animals in ecology in spite of large spatial separation, etc. Work is in progress along these lines which will be reported separately.

## Acknowledgments

The work of R.S., K.S. and M.L. has been supported by the Department of Science and Technology (DST), Government of India sponsored IRHPA research project. M.L. has also been supported by a DST Ramanna project and a DAE Raja Ramanna Fellowship. M.L. also acknowledges the support by the Alexander von Humboldt Foundation to visit PIK, where the work was completed. D.V.S. and J.K. acknowledge the support from EU under project No. 240763 PHOCUS(FP7-ICT-2009-C).

---

[1] I. Fischer, R. Vicente, J. M. Buldú, M. Peil, and C. R. Mirasso, Phys. Rev. Lett. **97**, 123902 (2006).  
[2] A. K. Engel, P. König, A. K. Kreiter, and W. Singer, Science **252**, 1177 (1991).  
[3] P. R. Roelfsema, A. K. Engel, P. König, and W. Singer, Nature **385**, 157 (1997).  
[4] D. Chawla, K. J. Friston, and D. Lumer, Neural Networks **14**, 727 (2001).  
[5] R. Vicente, L. L. Gollo, C. R. Mirasso, I. Fischer, and G. Pipa, Proc Natl Acad Sci **105**, 17157 (2008).  
[6] A. Englert, W. Kinzel, Y. Aviad, M. Butkovski, I. Reider, M. Zigzag, I. Kanter, and M. Rosenbluh, Phys. Rev.

Lett. **104**, 114102 (2010).

[7] L. Illing, C. D. Panda, and L. Shareshian, Phys. Rev. E **84**, 016213 (2011).

[8] A. Wagemakers, J. M. Buldú, and M. A. F. Sanjuán, Chaos **17**, 023128 (2007).

[9] R. Banerjee, D. Ghosh, E. Padmanaban, R. Ramaswamy, L. M. Pecora, and S. K. Dana, Phys. Rev. E **85**, 027201 (2012).

[10] A. S. Landsman, and I. B. Schwartz, Phys. Rev. E **75**, 026201 (2007).

[11] E. Klein, N. Gross, E. Kopelowitz, M. Rosenbluh, L. Khaykovich, W. Kinzel, and I. Kinter, Phys. Rev. E **74**, 046201 (2006).

[12] I. Kanter, E. Kopelowitz, and W. Kinzel, Phys. Rev. Lett. **101**, 084102 (2008).

[13] M. Lakshmanan, and D. V. Senthilkumar, *Dynamics of Nonlinear Time-Delay Systems* (Springer-Verlag, Berlin, 2011).

[14] J. N. Chiasson, and J. J. Loiseau, *Applications of Time Delay Systems* (Springer Verlag, Berlin, 2007).

[15] F. M. Atay, *Complex Time Delay Systems: Theory and Applications* (Springer-Verlag, Berlin, 2010).

[16] W-H. Kye, M. Choi, S. Rim, M. S. Kurdoglyan, C. H. Kim, and Y. J. Park, Phys. Rev. E **69**, 055202(R) (2004).

[17] C. Li, X. Liao, and K. W. Wong, Physica D **194**, 187 (2004).

[18] D. V. Senthilkumar, M. Lakshmanan, and J. Kurths, Phys. Rev. E. **74**, 035205(R) (2006).

[19] D. V. Senthilkumar, and M. Lakshmanan, Phys. Rev. E. **76**, 066210 (2007).

[20] D. V. Senthilkumar, K. Srinivasan, K. Murali, M. Lakshmanan, and J. Kurths, Phys. Rev. E. **82**, 065201(R) (2010).

[21] K. Srinivasan, D. V. Senthilkumar, K. Murali, M. Lakshmanan, and J. Kurths, Chaos **21**, 023119 (2011).

[22] A. Kittel, J. Parisi, and K. Pyragas, Physica D **112**, 459 (1998).

[23] M. de Sousa Vieira, Chaos **20**, 013131 (2010).

[24] K. Srinivasan, I. Raja Mohamed, K. Murali, and M. Lakshmanan, Int. J. Bifurcation and Chaos **21**, 725 (2011).

[25] J. Mulet, C. R. Mirasso, and I. Fischer, J. Opt.Soc. Am. B **6**, 97 (2004).

[26] L. B. Shaw, I. B. Schwartz, E. A. Rogers, and R. Roy, Chaos **16**, 015111 (2006).

[27] T. Pereira, M. S. Baptista, and J. Kurths, Phys. Rev. E **75**, 026216 (2007).

[28] M. Lakshmanan, and K. Murali, *Chaos in Nonlinear Oscillators: Controlling and Synchronization* (World Scientific, Singapore, 1995).

[29] M. C. Romano, M. Thiel, J. Kurths, I. Z. Kiss, and J. L. Hudson, Europhys. Lett. **71**, 466 (2005).

[30] N. Marwan, M. C. Romano, M. Thiel, and J. Kurths, Phys. Rep. **438**, 237 (2007).

[31] J. D. Farmer, Physica D **4**, 366 (1982).

[32] J. Kaplan, and J. Yorke, *Functional Differential Equations and Approximation of Fixed Points* (Edited by H. O. Peitgen and H. O. Walther, Springer, Berlin, Newyork, 1979).

[33] K. K. Rbrasovskii, *Stability of Motion* (Stanford University Press, Stanford, 1963).